NOTE ON THE STUDY OF MECHANICAL RESPONSE IN A PIEZOELECTRIC COMPOSITE TRANSDUCER

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(*Received October* 22, 1968; *Resubmitted January* 9, 1969)

ABSTRACT. The Laplace transform has been applied to calculate the mechanical response in a piezoelectric composite transducer with an elastic compliance, a part of which varies with time.

INTRODUCTION

The analysis of mechanical or electrical response in a piezoelectric transducer is important from the stand-point of generation and detection of ultrasonic waves. The studies in the nature of pulses, responses etc., have been discussed by Redwood (1961*a*, *b*) and Fillipzynaiski (1956). Sinha (1962*a*, *b*; 1965) and Giri (1966) have adopted the same method of solving problems. But all these studies are restricted to the problems in which the elastic compliance is assumed to be entirely invariable with regard to the time. It seems, therefore, worthwhile to extend the problem to the case of composite transducer in the sense used by Toulis (1963) with an elastic compliance which varies with time. The assumption of variation of elastic compliance is justified by the behaviour of electrets and also by the similarity that the electrets have with piezoelectric material (Mason 1948, 1950). In the present note the mechanical response in a composite-transducer has been calculated by using the Laplace transform as the tool.

FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

Let us consider a piezoelectric transducer in the form of a plate executing motion in thickness, in the direction of the x-axis. Let $x = 0$ and $x = X$ be the extremities, the portion $x = 0$ to $x = l$ is excited electromechanically. To the end $x = 0$ is applied electrical voltage V given by

$$V = V_0 t \tag{1.1}$$

where V_0 is constant. Obviously, this constitutes one of the type of composite transducer as suggested by Toulis (1963). Our object is to obtain mechanical response owing to an electrical input which varies with time.

The fundamental equations of the problem are naturally formed by the equations of elasticity, equations of Maxwell supplimented by the constitutive equations of material considered.

The equation of motion is

$$\rho \frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial T}{\partial x} \quad .. \quad (1.2)$$

where T, ζ are the mechanical stress and displacement in the x-direction, while ρ is the density of the material.

The divergence equation is given as

$$\frac{\partial D}{\partial x} = 0 \quad .. \quad (1.3)$$

The constitutive equations of material as in Mason (1948, 1950) are

$$T = C_{11} \frac{\partial \zeta}{\partial x} - hD \quad .. \quad (1.4)$$

$$E = -h \frac{\partial \zeta}{\partial x} + \frac{D}{\epsilon} \quad .. \quad (1.5)$$

where C_{11} is the elastic compliance, h the piezoelectric constant ϵ the electric permittivity and E the electric intensity. The elastic compliance C_{11} is depending on time which can be written as

$$C_{11} = C_0 + C_1 \frac{\partial}{\partial t} \quad .. \quad (1.6)$$

A differential equation for ζ is obtained from the equations (1.2), (1.3), (1.4) using the relation (1.6). Taking the Laplace transform of this equation, we get

$$\frac{\partial^2 \zeta}{\partial x^2} - \frac{\rho}{C_0 + C_1} p^2 \bar{\zeta} = 0 \quad .. \quad (1.7)$$

Solving (1.7), we have

$$\bar{\zeta} = A \exp\left(-p\sqrt{\frac{\rho}{C_0 + C_1 p}}\right) x + B \exp\left(+p\sqrt{\frac{\rho}{C_0 + C_1 p}}\right) x \quad .. \quad (1.8)$$

for $0 < x < l$

$$\bar{\zeta}' = A' \exp\left(-p\sqrt{\frac{\rho}{C_0 + C_1 p}}\right) x + B' \exp\left(p\sqrt{\frac{\rho}{C_0 + C_1 p}}\right) x \quad .. \quad (1.9)$$

for $l < x < X$

To evaluate constants A and B we have an important relation adopted by Redwood (1961*a*, *b*)

$$\bar{F} = p\sqrt{C_0+C_1p}\,.\,Z\left[-A\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)x+B\exp\left(p\sqrt{\frac{\rho}{C_0C+_1p}}\right)x\right]$$

$$-h\bar{Q} \qquad \text{.. (1.10)}$$

Also, we have

$$\bar{V} = -h[(\bar{\zeta})_l-(\bar{\zeta})_0]+\frac{\bar{Q}}{C'_0} \qquad \text{.. (1.11)}$$

where $\bar{V}$, $\bar{Q}$ are the Laplace transforms of charge Q and voltage V and C'_0 is the static capacitance, We assume the transducer of impedance Z_c to be connected at extremities with the transducers of impedances Z_1 and Z_2. We specify the constants and entities for the mechanical systems attached to the transducer at $x = 0$ by subscript 1 and that at $x = X$ by subscript 2. The conditions of continuity of the force and displacement at the extremities $x = 0$, at junction $x = l$, and at $x = X$, when formulated give rise to

(a) at $x = 0$

$$(\bar{F}_1)_0 = (F)_0$$

$$(\bar{\zeta}_1)_0 = (\bar{\zeta})_0$$

(b) at $x = l$

$$(\bar{F}) = (\bar{F}')$$

$$(\bar{\zeta}) = (\bar{\zeta}')$$

(c) at $x = X$

$$(\bar{F}') = (\bar{F}_2)$$

$$(\bar{\zeta}') = (\bar{\zeta}_2) \qquad \text{.. (1.12)}$$

SOLUTION OF THE PROBLEM

To simplify the calculations we consider a rigidly backed plate transducer (2.12) give rise to

$$p\sqrt{C_0+C_1p.}\,Z_1B_1 = p\sqrt{C_0+C_1p}.\,Z_c(-A+B)-h\bar{Q} \qquad \text{.. (2.1)}$$

$$B_1 = A+B \qquad \text{.. (2.2)}$$

$$-Z_cA\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l+Z_cB\exp\left(p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l=$$

$$-A'\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l+B'\exp\left(p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l \quad .. \quad (2.3)$$

$$A\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l+B\exp\left(p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l. = A'\exp\left(\frac{\rho}{C_0+C_1p}\right)l$$

$$+B'\exp\left(p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l \quad .. \quad (2.4)$$

$$A'\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)\chi+B'\exp\left(p\sqrt{\frac{\rho}{C_0+C_1p}}\right)\chi=0 \quad .. \quad (2.5)$$

Substituting the values of $(\bar{\zeta})$, $(\bar{\zeta})_0$ and $\bar{Q}$ in the equation (2.11) we get

$$\bar{V}=hA-hB-hA'\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l-hB'\exp\left(p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l$$

$$+\frac{B}{C'_0h}[p\sqrt{C_0+C_1p}(Z_e-Z_1)]-\frac{A}{C'_0h}[p\sqrt{C_0+C_1p}(Z_c+Z_1)] \quad .. \quad (2.6)$$

The equations (2.1), (2.2), (2.5) and (2.6) help us to express A', B' in terms of A and B. Putting these values of A', B' in (2.3) and (2.4), we get two linear equations in A and B. Solving these two equations we get

$$A=-\frac{\bar{V}}{h[K_2K_3+K_1K_4]}\left[K_4+K_2\frac{Z'_c}{Z_c}\left\{\frac{1+\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)}{1-\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)}\right\}\right] \quad .. \quad (2.7)$$

$$B=\frac{\bar{V}}{h[K_2K_3+K_1K_4]}\left[K_1\frac{Z'_c}{Z_c}\left\{\frac{1+\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)}{1-\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)}\right\}-K_3\right] \quad .. \quad (2.8)$$

where

$$K_1=\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l-(1-R_1p\sqrt{C_0+C_1p})$$

$$K_2 = \exp\left(p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l-(1-R_1p\sqrt{C_0+C_1p}).$$

$$K_3 = \exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l-\frac{Z'_c}{Z_c}\frac{1+\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)}{1-\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)}\times$$

$$(1-\exp\sqrt{C_0+C_{1p}})$$

$$K_4 = \exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l+\frac{Z'_c}{Z_c}\frac{1+\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)}{1-\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)}\times$$

$$(1-R_2p\sqrt{C_0+C_1p})$$

and

$$R_1 = \frac{Z'_c+Z_1}{C_0'h^2}, \qquad R_2 = \frac{Z'_c-Z_c}{C_0'h^2}$$

From (1.8) we have

$$(\zeta)_0 = A+B. \qquad .. \quad (2.9)$$

Putting the values of A and B in (3.13) we get

$$(\bar{\zeta})_0 = \frac{\bar{V}}{h[K_2K_3+K_1K_4]}\left[(K_1-K_2).\frac{Z'_c}{Z_c}\left\{\frac{1+\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)}{1-\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)}\right\}\right.$$

$$\left.-(K_3+K_4)\right] \qquad .. \quad (2.10)$$

The values of (K_1-K_2), (K_3+K_4), $(K_2K_3+K_1K_4)$ are obtained as

$$(K_1-K_2) = \exp\left(p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l\left[(R_1-R_2)(p\sqrt{C_0+C_1p})\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l\right.$$

$$\left.-\left\{1-\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l\right\}\right]$$

$$(K_3+K_4) = \exp\left(p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l\left[1+\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l\right.$$

$$+\frac{Z'_e}{Z_e}\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l\left\{1+2\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)\right\}$$

$$\{(R_1-R_2)(p\sqrt{C_0+C_1p})\}\Big]$$

$$K_2K_3+K_1K_4=\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l\left[1-R_1p\sqrt{C_0+C_1p}\left(1+\frac{Z'_e}{Z_e}\right)-\right.$$

$$2\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l+(1+R_2p\sqrt{C_0+C_1p})\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l.$$

$$-\frac{Z'_e}{Z_e}(1-R_2p\sqrt{C_0+C_1p})\left\{1+2\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)\right\}$$

$$\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l+2\frac{Z'_e}{Z_e}(1-R_1p\sqrt{C_0+C_1p})\exp\left(2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)$$

Hence from (1.14), we get

$$(\bar{\zeta})_0=\frac{\bar{V}}{h(1-R_1p\sqrt{C_0+C_1p})\left(1+\frac{Z'_e}{Z_e}\right)}\left[1-\frac{1}{(R_1p\sqrt{C_0+C_1p})\left(1+\frac{Z'_e}{Z_e}\right)}\right.$$

$$2\exp\left(-p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l-(1-R_2p\sqrt{C_0+C_1p})\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l$$

$$+\frac{Z'_e}{Z_e}(1-R_2p\sqrt{C_0+C_1p})\left\{1+2\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)\right\}$$

$$\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l-2\frac{Z'_e}{Z_e}(1-R_1p\sqrt{C_0+C_1p})$$

$$\left.\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)\right]^{-1}\left[1+\frac{Z'_e}{Z_e}+\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)l\right.$$

$$\left\{1-\frac{Z'_e}{Z_e}\left(1+\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)+2\frac{Z'_e}{Z_e}\right.\right.$$

$$\left.\exp\left(-2p\sqrt{\frac{\rho}{C_0+C_1p}}\right)(X-l)\right] \quad \text{.. (2.11)}$$

To get its inverse transform, following Redwood, we expand relevant terms of right hand side of (2.11) binomially and by considering first order term in the expansion

$$(\bar{\zeta})_0 = \frac{\bar{V}}{h[1-\bar{R}_1 p\sqrt{C_0}+C_1 \bar{p}]}$$

From (1.1) we obtain

$$V = \frac{V_0}{p^2}$$

$$(\zeta)_0 = \frac{V_0}{h} t + \frac{V_0 R\sqrt{C_0}}{h} - \tfrac{1}{2}\frac{V_0 R_1 C_1}{\sqrt{C_0}h}\delta(t)$$

where $\delta(t)$ is Dirac's delta function.

It is clear from the above expression that subject to the first order of approximation, the mechanical response emitted by a composite transducer consists of three parts, first varies with time, second is steady and third is impulsive.

In the similar way the mechanical response may be obtained upto the second order of approximation.

The author is thankful to Dr. R. R. Giri, Jadavpur University, for his help and guidance in the preparation of this paper.

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